The sum of two measurable functions

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December 22, 2005*

Summary

Following Weizsäcker [3], we use this notation: For a complete probability space $(\Omega, \Sigma, \mathbf{P})$ and a locally convex space E, denote by $L^0(\Omega, \Sigma, \mathbf{P}, E)$ the set of all Borel-measurable functions $f : \Omega \to E$ for which the image measure $f[\mathbf{P}]$ on E is Radon.

In 1976 E. Thomas asked, in a conversation with the author, whether $L^0(\Omega, \Sigma, \mathbf{P}, E)$ is always closed under addition. The question is motivated by the observation that some of the results in [2] can be proved for functions in $L^0(\Omega, \Sigma, \mathbf{P}, E)$.

This note presents an example where $L^0(\Omega, \Sigma, \mathbf{P}, E)$ is <u>not</u> closed under addition. However, Weizsäcker [3] showed that this obstacle is not as serious as would seem.

Terminology

All measures will be probability measures, i.e. positive and with total mass 1. Say that $(X, \mathcal{A}, \mu)$ is a compact Radon measure space if X is a compact Hausdorff space, $\mathcal{A}$ is a sigma-algebra on X containing all Borel subsets of X and μ is a complete measure on $\mathcal{A}$ such that

$$\mu B \;=\; \sup \{\; \mu K \mid K \subseteq B \;\text{ and }\; K \;\text{ is compact}\; \}$$

for every $B \in \mathcal{A}$.

When $(X, \mathcal{A})$ and $(B, \mathcal{B})$ are two measurable spaces (sets with sigma-algebras), denote by $\mathcal{A} \otimes \mathcal{B}$ the product sigma-algebra on $X \times Y$; this is the smallest sigma-algebra on $X \times Y$ making both projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ measurable.

*Transcribed from the author's manuscript dated April 1980.

Example

The example will be constructed in three steps.

Step 1 *Construct a compact Radon measure space* $(X, \mathcal{A}, \mu)$ *such that* $\mu B = 0$ *whenever* $B \in \mathcal{A}$ *has cardinality less than or equal to* $2^{\aleph_0}$*.*

Construction Let I be a set of cardinality $2^{\aleph_0}$, let X be the compact space $\{0,1\}^I$, and let μ be the standard product measure on X (defined on $\mathcal{A}$, the μ-completion of the Borel sigma-algebra in X). That is, μ is the product of measures on $\{0,1\}$ each of which gives measure $\frac{1}{2}$ to $\{0\}$ and $\frac{1}{2}$ to $\{1\}$.

For every subset J of I, define an automorphism T_J of $(X, \mathcal{A}, \mu)$ by

$$T_J(\{x_i\}_{i \in I}) \;=\; \{y_i\}_{i \in I}$$

where $y_i = x_i$ for $i \in J$ and $y_i = 1 - x_i$ for $i \in I \backslash J$.

If $B \in \mathcal{A}$ has cardinality $\leq 2^{\aleph_0}$ then there is a set $J \subseteq I$ such that $B \cap T_J(B) = \emptyset$. Indeed, choose an injective map $\alpha : B \times B \to I$ and define

$$J \;=\; \{\; j \in I \mid j = \alpha(\{x_i\}_{i \in I}, \{y_i\}_{i \in I}) \;\text{ for }\; \{x_i\}, \{y_i\} \in B \;\text{ and }\; x_j \neq y_j \;\}\,.$$

It follows that for each $B \in \mathcal{A}$ of cardinality $\leq 2^{\aleph_0}$ there is a sequence of μ-automorphisms $S_1, S_2, S_3, \ldots$ such that

$$B_k \cap S_k(B_k) = \emptyset\,, k = 1, 2, 3, \ldots\,,$$

where

$$B_1 = B\,, \;\; B_{k+1} = B_k \cup S_k(B_k)\,.$$

Hence there are infinitely many disjoint sets of the same measure as B. Therefore $\mu B = 0$.

Step 2 *Construct a compact Radon measure space* $(X, \mathcal{A}, \mu)$ *and a measure* ν *on the product sigma-algebra* $\mathcal{A} \otimes \mathcal{A}$ *such that for the "diagonal"* $D = \{(x,x) | x \in X\}$ *and the projections* $\pi_1 : X \times X \to X$ *and* $\pi_2 : X \times X \to X$ *we have*

(i) $\nu G = 1$ *for every* $G \in \mathcal{A} \otimes \mathcal{A}$ *such that* $G \cup D = X \times X$*,*
and $\nu H = 1$ *for every* $H \in \mathcal{A} \otimes \mathcal{A}$ *such that* $H \supseteq D$*;*

(ii) $\pi_1[\nu] = \mu = \pi_2[\nu]$*.*

Construction Take the $(X, \mathcal{A}, \mu)$ constructed in Step 1. Denote by $\beta : X \to X \times X$ the map defined by $\beta(x) = (x, x)$. We have $\beta^{-1}(G) \in \mathcal{A}$ for each $G \in \mathcal{A} \otimes \mathcal{A}$; let $\nu G = \mu(\beta^{-1}(G))$ for $G \in \mathcal{A} \otimes \mathcal{A}$ (that is, $\nu = \beta[\mu]$). Since both $\pi_1 \circ \beta$ and $\pi_2 \circ \beta$ are the identity map on X, (ii) follows.

If $H \in \mathcal{A} \otimes \mathcal{A}$ and $H \supseteq D$ then, by the definition of ν, we have $\nu H = 1$.

Take $G \in \mathcal{A} \otimes \mathcal{A}$ such that $G \cup D = X \times X$. We have

$$\nu G \;=\; \inf \left\{ \sum_{n=1}^{\infty} \nu(B_n \times C_n) \;\middle|\; B_n\,, C_n \in \mathcal{A} \;\text{ and }\; \bigcup_{n=1}^{\infty} (B_n \times C_n) \supseteq G \right\}$$

(see e.g. [1], 13.A); thus it suffices to show that

$$\sum_{n=1}^{\infty} \nu(B_n \times C_n) \geq 1 \;\text{ whenever }\; \bigcup_{n=1}^{\infty} (B_n \times C_n) \supseteq G\,, \;\; B_n, C_n \in \mathcal{A}.$$

Fix such B_n, C_n and let $V = \bigcup_{n=1}^{\infty} (B_n \times C_n)$. Then $(X \times X) \backslash V \subseteq D$. We show that the cardinality of $(X \times X) \backslash V$ is at most $2^{\aleph_0}$: If $(x, x), (y, y) \in (X \times X) \backslash V$ and $x \neq y$ then there is n such that $(x, y) \in B_n \times C_n$ and $(x, x) \notin B_n \times C_n$; hence x and y are separated by C_n. It follows that $(X \times X) \backslash V$ has at most $2^{\aleph_0}$ points. Consequently, $\beta^{-1}((X \times X) \backslash V)$ has at most $2^{\aleph_0}$ points and

$$\nu((X \times X) \backslash V) \;=\; \mu(\beta^{-1}((X \times X) \backslash V) = 0$$

by the property of μ. Thus

$$\sum_{n=1}^{\infty} \nu(B_n \times C_n) \geq \nu V \;=\; \nu V + \nu((X \times X) \backslash V) \;=\; \nu(X \times X) \;=\; 1\,.$$

It follows that $\nu G = 1$.

Step 3 *Construct a complete probability space* $(\Omega, \Sigma, \mathbf{P})$*, a locally convex space* E *and two functions* $f, g : \Omega \to E$ *such that*

(a) $f^{-1}(B), g^{-1}(B) \in \Sigma$ *for every Borel set* $B \subseteq E$*;*

(b) the image measures $f[\mathbf{P}]$ *and* $g[\mathbf{P}]$ *are Radon;*

(c) the function $h = f + g$ *has the property* $h^{-1}(0) \notin \Sigma$.

Thus, in this example, $L^0(\Omega, \Sigma, \mathbf{P}, E)$ is not closed under addition.

Construction Take the $(X, \mathcal{A}, \mu)$, ν, D, π_1 and π_2 as in Step 2.

Every compact Hausdorff space Y is a topological subspace of a locally convex space (e.g. let $C(Y)$ be the Banach space of real-valued continuous functions on Y; then Y embeds canonically into the dual of $C(Y)$ endowed with the w* topology). Fix such an embedding $e : X \hookrightarrow E$ of X into a suitable locally convex space E. Let $(\Omega, \Sigma, \mathbf{P})$ be the completion of $(X \times X, \mathcal{A} \otimes \mathcal{A}, \nu)$, and let $f = e \circ \pi_1$, $g = -\, e \circ \pi_2$.

Now (a) is obvious, and (b) is true because the measures $f[\mathbf{P}]$ and $g[\mathbf{P}]$ are continuous images of the Radon measure μ (by (ii) in Step 2). Finally, $h^{-1}(0) = D$ and $D \notin \Sigma$ in view of (i) in Step 2; that proves (c).

References

[1] P.R. Halmos. *Measure Theory*. Springer-Verlag 1974.

[2] G.E.F. Thomas. *Integration of functions with values in locally convex Suslin spaces.* Trans. Amer. Math. Soc. 212 (1975) 61-81.

[3] H. v. Weizsäcker. *Strong measurability, liftings and the Choquet-Edgar theorem.* Vector Space Measures and Applications II. Springer-Verlag Lecture Notes in Mathematics Vol. 645 (1977) 209-218.